

Entangled black holes as ciphers of hidden information

Samuel L. Braunstein

Computer Science, University of York, York YO10 5DD, UK

Hans-Jürgen Sommers

Theoretische Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany

Karol Życzkowski

*Institute of Physics, Jagiellonian University, 30-059 Krakow, Poland and
Center for Theoretical Physics, Polish Academy of Science, 02-668 Warszawa, Poland*

(Dated: July 4, 2009)

The black-hole information paradox has fueled a fascinating effort to reconcile the predictions of general relativity and those of quantum mechanics. Gravitational considerations teach us that black holes must trap everything that falls into them. Quantum mechanically the mass of a black hole leaks away as featureless (Hawking) radiation. However, if Hawking's analysis turned out to be accurate then the information would be irretrievably lost and a fundamental axiom of quantum mechanics, that of unitary evolution, would likewise fail. Here we show that the information about the matter that collapses to form a black hole becomes encoded into pure correlations within a tripartite quantum system, the quantum analog of a one-time pad until very late in the evaporation, provided we accept the view that the thermodynamic entropy of a black hole is due to entropy of entanglement. In this view the black hole entropy is primarily due to trans-event horizon entanglement between external modes neighboring the black hole and internal degrees of freedom of the black hole.

A powerful tool for studying black hole evaporation as a unitary process is in terms of random subsystems. The starting point of this approach is that the evaporative dynamics can be modeled by sampling a random subspace from the black hole interior, of dimensionality equaling the radiation subsystem. This idea was originally formulated [1] in a model where all the in-falling matter was in a pure state $|i\rangle$, so

$$|i\rangle_{\text{int}} \rightarrow (U|i\rangle)_{\text{RB}}. \quad (1)$$

Here the initial internal (int) Hilbert space of the black hole may be thought of as evolving under a random unitary U followed by the 'emission' of radiation into (a now randomly selected) subspace R with the *reduced-size* interior labeled B . A key assumption of any random matrix calculation is the dimensionality of the space on which the random matrices act. For black hole evaporation it was argued [1] that this dimensionality should be well approximated from the *thermodynamic* entropy $S_{\text{BH}} = \mathcal{A}/4$ of a black hole of area \mathcal{A} , giving a dimensionality $\dim(\text{int}) = RB = e^{S_{\text{BH}}}$ — where we reuse subspace labels for Hilbert space dimensionalities. Therefore, the black hole interior comprises $n = \log_2[\dim(\text{int})]$ qubits.

Within this original model, 'discernable information' is defined as the deficit from maximal entropy in the radiation subspace. It was found that this quantity remains almost zero, until half the qubits of the entire black hole had been radiated, after which the discernable information rises at the rate of roughly two bits for every qubit radiated [1]. This behavior suggests that first entanglement is created, followed by dense coding [2] of *classical* information about the initial state. By entangling the

state of the in-fallen matter in this model with some distant reference (ref) subspace, one can track the flow of quantum information using information theoretic tools. In this way Eq. (1) becomes [3]

$$\frac{1}{\sqrt{K}} \sum_{i=1}^K |i\rangle_{\text{ref}} \otimes |i\rangle_{\text{int}} \rightarrow \frac{1}{\sqrt{K}} \sum_{i=1}^K |i\rangle_{\text{ref}} \otimes (U|i\rangle)_{\text{RB}}. \quad (2)$$

Here $k = \log_2 K$ is the number of qubits describing the quantum state of the matter used to form the black hole. That this is tiny in comparison to the number of qubits comprising the black hole itself, $k \ll n$, is one of the signature properties of a black hole [4]. Using the decoupling theorem [5] (see Appendix for details), we may conclude that prior to $\frac{1}{2}(n-k) - c$ qubits having been radiated, the quantum information about the in-fallen matter is encoded within the black hole interior with fidelity at least $1 - 2^{-c}$; whereas after a further $k+2c$ qubits have been radiated, the in-fallen matter is encoded within the radiation subspace with fidelity at least $1 - 2^{-c}$ (see also Ref. 3 for this latter result). The quantum information about the in-fallen matter leaves in a narrow 'pulse' at the radiation emission rate.

Our key point of departure from previous work is motivated by the well-accepted result from many-body quantum theory that the entanglement of across a boundary will generically scale as the boundary's area [6]. It is therefore natural to conjecture that a black hole's thermodynamic entropy might be primarily due to entropy of entanglement between modes external to, but in the neighborhood of the event horizon and modes of the black hole's interior. Such a conjecture has indeed been made [7, 8, 9, 10, 11, 12]; it holds naturally for some models

of black holes [10, 11] and even resolves some difficulties associated with computing entropy at the microscopic level [12]. Here we show that this conjecture leads to a radically different picture of information flow from black holes.

As with Ref. 3 we tag the information about the matter that collapsed to form the black hole by entanglement with some distant reference (ref) subsystem. If we assume that there is *no* so-called “bleaching” mechanism which can strip away all or part of the information about the in-fallen matter as it passes the event horizon, then the initial quantum state of the black hole interior (int) and its surroundings has the unique form [13]

$$\frac{1}{\sqrt{K}} \sum_{i=1}^K |i\rangle_{\text{ref}} \otimes \sum_j \sqrt{p_j} (|i\rangle \otimes |j\rangle \oplus 0)_{\text{int}} \otimes |j\rangle_{\text{ext}}, \quad (3a)$$

up to overall int-local and ext-local unitaries. Here $\oplus 0$ means we pad any unused dimensions of the interior space by zero vectors [13] and $\rho_{\text{ext}} = \sum_j p_j |j\rangle_{\text{ext}} \langle j|$ is the reduced density matrix for the external (ext) neighborhood modes. Again we take the dimension of the interior space as $\dim(\text{int}) = RB = e^{S_{\text{BH}}}$.

As with earlier work [1, 3], described by Eqs. (1) and (2), we apply a random unitary, constrained by causality and acting solely on the black hole interior, $|\psi\rangle_{\text{int}} \rightarrow (U|\psi\rangle)_{RB}$ to randomly sample the radiation subspace (R). Eq. (3a) then becomes

$$\rightarrow \frac{1}{\sqrt{K}} \sum_{i=1}^K |i\rangle_{\text{ref}} \otimes \sum_j \sqrt{p_j} [U(|i\rangle \otimes |j\rangle \oplus 0)]_{RB} \otimes |j\rangle_{\text{ext}}. \quad (3b)$$

It will be convenient to define

$$x \equiv \log_2(RB/K) + \log_2(\text{tr } \rho_{\text{ext}}^2), \quad (4)$$

which roughly quantifies the number of excess unentangled qubits within the initial encoding of the black hole in Eq. (3a). Note that $0 \leq x \leq \log_2(RB/K)$.

Again using the distant reference to tag the information (see Appendix for details), it is easy to see that for all but the final $k + \frac{1}{2}x + c$ qubits radiated, the information about the in-fallen matter is encoded in the combined space of external neighborhood modes and black hole interior with fidelity at least $1 - 2^{-c}$. Similarly, for all but the initial $k + \frac{1}{2}x + c$ qubits radiated, this information is encoded in the combined radiation and external neighborhood modes with fidelity at least $1 - 2^{-c}$. In addition, at all times this information is encoded with unit fidelity within the joint radiation and interior subspaces.

In other words, between the initial and final $k + \frac{1}{2}x + c$ qubits radiated, the information about the in-fallen matter is effectively deleted from each subsystem individually [13, 14], instead being encoded in any two of the three of subsystems. During this time, the information

about the in-fallen matter is to an excellent approximation encoded within the perfect correlations of a quantum one-time pad [13, 15] consisting of the three subsystems: the radiation, the external neighborhood modes, and the black hole interior. This description applies to the entire evaporation period except for the short encoding and decoding periods (assuming small x above). A heuristic picture showing a smooth flow of information is given in Appendix .

We now consider what happens if additional matter is dumped into the black hole after its creation. Following Ref. 3, we model this process via cascaded random unitaries on the black hole interior — one unitary before each radiated qubit. Within the pure state model of Eq. (2), it was argued [3] that after half of the initial qubits had radiated away, any information about matter subsequently falling into the black hole would be “reflected” immediately at roughly the radiation emission rate [3]. A subtle flaw to this argument is due to the omission of the fact that a black hole’s entropy is non-extensive, typically scaling as the square of the black hole’s mass M^2 : for every q qubits dumped into a black hole, the entropy increases by $O(qM) \gg q$. Likewise, the number of unentangled qubits within the black hole will increase by $O(qM)$. Therefore, within the cascaded unitary pure-state model, the reflection described in Ref. 3 would not begin immediately, but only after a large delay in time of $O(qM^2)$. Notwithstanding the delay, the very different *behaviors* of the black hole in the first and second halves of its life endows it with a kind of quasistatic “hair” associated with its history since creation.

By contrast, the one-time-pad description of evaporation [for black holes described by the entropy-as-entanglement conjecture of Eq. (3)] paints a very different picture. Instead of the reflection of information found in the pure-state model, here, any additional qubits thrown into the black hole will immediately begin to be encoded into the tripartite correlation structure (assuming negligible x). Therefore, just as in the uncascaded case, the decoding into the radiation subspace of *all* the information about all the in-fallen matter will only occur at the very end of the evaporation. The non-extensive increase in black hole entropy is taken up as entanglement with external neighborhood modes so no further delays occur. Importantly, entanglement-based black holes really are “hairless”: their *behavior* does not qualitatively change in time.

As we have seen, in the pure-state model of a black hole the information about the in-fallen matter leaves in a narrow pulse after half the qubits have evaporated, whereas for the entangled-state model, this information appears in the out-going radiation only at the end of the evaporation. One way to reconcile these two models is if the pure-state model were run for twice as many qubits, but stopped just after the information about the in-fallen matter had escaped. If we accepted the model

of a black hole as highly entangled across its event horizon we could justify this reconciliation as a rough approximation: In particular, instead of fixing a boundary at the event horizon we could fix it somewhat further out, say at $r = 3M$. In this case, the dimensionality enclosed would be roughly the square of that of the black hole interior space itself by including the contribution from the external neighborhood modes; this would therefore yield roughly twice as many qubits as the black hole itself holds. The trans-boundary entanglement at $r = 3M$ would be approximately static over the entire course of evaporation so it could be factored out as non-dynamical, thus leaving an approximately pure-state description. Finally, once the original number of qubits had evaporated away (now half the total for our modified pure-state model) the black hole interior would be exhausted of Hilbert space and evaporation would cease. This suggests that despite the incompatibility between the two models, a pure-state analysis, if properly set up, can capture important features of information retrieval from the entangled-state model.

Recently, the *no-hiding theorem* [13, 14] was used to prove that Hawking’s prediction of featureless radiation implied that the information about the in-fallen matter could not be in the radiation field, but must reside in the remainder of Hilbert space — then presumed to be the black hole interior. That work presented a strong form of the black hole information paradox pitting the predictions of general relativity against those of quantum mechanics [13]. Here we have shown that trans-event horizon entanglement provides a way out, since now the “remainder of Hilbert space” comprises both the black hole interior and external neighborhood modes. Because the evaporating black hole actually involves three subsystems, the information may be encoded within them as pure correlations via a quantum one-time pad [13, 15]: the information is in principle retrievable from any two of the three subsystems, yet inaccessible from any single subsystem alone. This simultaneous encoding of information externally (in the combined radiation and external neighborhood modes) and ‘internally’ (if one *stretches* the horizon to envelope the bulk of the external neighborhood modes in addition to the black hole interior) is reminiscent of Susskind’s principle of black hole complementarity [16]. Yet, the overlap between the interior and exterior in this picture eliminates any need for a temporary or unobservable violation of the no-cloning theorem. Within the one-time pad encoding trans-event horizon entanglement provides a mechanism whereby Hawking’s calculations may accurately describe the behavior of outgoing radiation from a black hole until very late in its evaporation; it does not necessarily solve the paradox, but it delays for as long as possible the clash between two of our most cherished and fundamental theories of nature.

SLB gratefully acknowledges the hospitality of the Abe Watssman Institute for Innovative Thinking, where this work was initiated. HJS and KZ acknowledge financial support by the Deutsche Forschungsgemeinschaft under the project SFB/TR12 and grant number DFG-SFB/38/2007 of the Polish Ministry of Science and Higher Education.

APPENDIX A

We now summarize the decoupling theorem [5]. Consider a pure state tripartite $|\Psi\rangle_{\text{ref,ext},A_1A_2}$, where the joint subsystems A_1A_2 will be decomposed as either the radiation modes and interior black holes modes RB or vice-versa BR . Tracing out the external neighborhood modes gives

$$\rho_{\text{ref},A_1A_2} \equiv \text{tr}_{\text{ext}}(|\Psi\rangle_{\text{ref,ext},A_1A_2}\langle\Psi|). \quad (5)$$

Next, consider a (random) unitary applied to the joint subsystems A_1A_2 . This allows us to define

$$\sigma_{\text{ref},A_2}^U \equiv \text{tr}_{A_1}(U_{A_1A_2} \rho_{\text{ref},A_1A_2} U_{A_1A_2}^\dagger). \quad (6)$$

Then the decoupling theorem [5] states that

$$\begin{aligned} & \left(\int_{U \in U(A_1A_2)} dU \left\| \sigma_{\text{ref},A_2}^U - \sigma_{\text{ref}}^U \otimes \sigma_{A_2}^U \right\|_1 \right)^2 \\ & \leq \frac{A_2 K}{A_1} (\text{tr} \rho_{\text{ref},A_1A_2}^2 + \text{tr} \rho_{\text{ref}}^2 \text{tr} \rho_{A_1A_2}^2), \end{aligned} \quad (7)$$

where states with ‘missing’ subscripts denote further tracing out of the relevant subspaces. Now $1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1$, where the trace norm is defined by $\|X\|_1 \equiv \text{tr}|X|$ and the fidelity by $F(\rho, \sigma) \equiv \|\sqrt{\rho}\sqrt{\sigma}\|_1$. As a consequence, the fidelity with which the entangled state describing the in-fallen matter is encoded within the combined $\text{ref}, A_1, \text{ext}$ subsystem is bounded below by [13]

$$1 - \left(\frac{A_2 K}{A_1} \text{tr} \rho_{\text{ext}}^2 \right)^{\frac{1}{2}}, \quad (8)$$

where we use the fact that $\text{tr} \rho_{\text{ext}}^2 = \text{tr} \rho_{\text{ref},A_1A_2}^2 \geq \text{tr} \rho_{\text{ref}}^2 \text{tr} \rho_{A_1A_2}^2$ for our model. Eq. (8) expresses nothing more than the lower bound to the fidelity with which the quantum state about the in-fallen matter may be theoretically reconstructed from this joint subspace. We note that the results for the pure-state model of black hole evaporation [3] may be recovered by setting $\text{tr} \rho_{\text{ext}}^2 = 1$.

APPENDIX B

The rigorous results from the manuscript may be heuristically visualized by following how the correlations

with the distant reference system behave. For a pure tripartite state XYZ , these correlations satisfy

$$C(X:Y) + C(X:Z) = S(X), \quad (9)$$

Here $S(X)$ is the von Neumann entropy for subsystem X and $C(X:Y) \equiv \frac{1}{2}[S(X) + S(Y) - S(X,Y)]$, one-half the quantum mutual information, is a measure of correlations between subsystems X and Y . Relation (9) is additive for a pure tripartite state, so the correlations with subsystem X smoothly move from subsystems Y to Z and vice-versa.

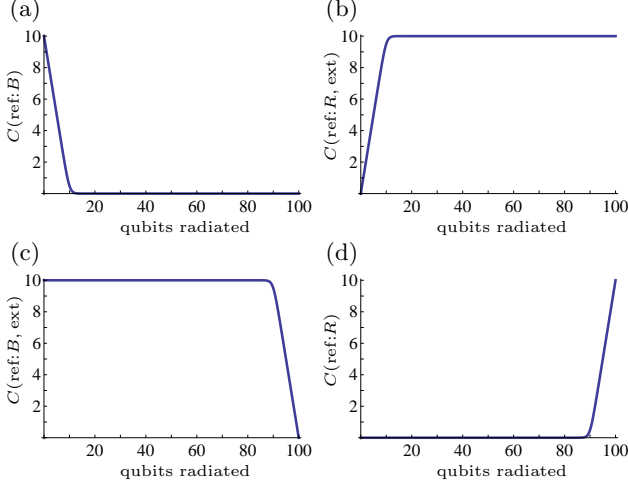


FIG. 1: Correlations to the reference subsystem as a function of the number of qubits radiated ($\log_2 R$). Correlations between the reference (ref) subsystem and: (a) black hole interior, B ; (b) radiation, A , and external (ext) neighborhood modes; (c) black hole interior and external neighborhood modes; and (d) radiation alone. Note that, as expected from Eq. (9), the sum of C 's in subplots (a) and (b) is a constant, as is that of subplots (c) and (d). In each subplot, the in-fallen matter consists of $k = 10$ qubits and the black hole initially consists of $\log_2 RB = 100$ qubits with $x = 0$. (Entropies are evaluated using base-two logarithms.)

For simplicity, in this Appendix we restrict ourselves to the case where

$$\rho_{\text{ext}} = \frac{1}{N} \sum_{j=1}^N |j\rangle_{\text{ext}} \langle j|, \quad (10)$$

and where $x = 0$. We computed the above measure of correlations, Eq. (9), from von Neumann entropies approximated using the average purity (see Appendix for details); numerical calculations showed this as a good approximation for systems of even a few qubits. Fig. 1 shows a typical scenario (assuming no excess unentangled qubits): A black hole is assumed to be created from in-fallen matter comprising k qubits of information and negligible excess unentangled qubits. Within the first k qubits radiated, information about the in-fallen matter (a) vanishes from the black hole interior at roughly the

radiation emission rate and (b) appears in the joint radiation and external neighborhood subspace. From then until just before the final k qubits are radiated, the in-fallen matter's information is encoded in a tripartite state, involving the radiation, external neighborhood and interior subspaces, subplots (b) and (c). In the final k qubits radiated the information about the in-fallen matter is released from its correlations and appears in the radiation subsystem alone, subplot (d). This qualitative picture is in excellent agreement with the results from the decoupling theorem and its generalization.

In order to approximate the computation of the correlation measure described in the text, we use a lower bound for a subsystem with density matrix ρ

$$\langle\langle S(\rho) \rangle\rangle \geq -\langle\langle \ln p(\rho) \rangle\rangle \geq -\ln \langle\langle p(\rho) \rangle\rangle. \quad (11)$$

Here $S(\rho) = -\text{tr } \rho \ln \rho$ is the von Neumann entropy of ρ , $p(\rho) = \text{tr } \rho^2$ is its purity, and here $\langle\langle \dots \rangle\rangle$ denotes averaging over random unitaries with the Haar measure. The former inequality above is a consequence of the fact that the Rényi entropy is a non-increasing function of its argument [17], and the latter follows from the concavity of the logarithm and Jensen's inequality. We may estimate the von Neumann entropies required then by the rather crude approximation $\langle\langle S(\rho) \rangle\rangle \approx -\ln \langle\langle p(\rho) \rangle\rangle$, which turns out to be quite reasonable for spaces with even a few qubits.

Although traditional methods [18] may be used to compute these purities, a much simpler approach is to use the approach from Ref. 5. In particular, for a typical purity of interest we use the following decomposition

$$\begin{aligned} \text{tr } \sigma_{R,\text{ext}}^{U^2} &= \text{tr}(\sigma_{R,\text{ext}}^U \otimes \sigma_{R',\text{ext}'}^U \mathcal{S}_{R,\text{ext};R',\text{ext}'}) \\ &= \text{tr}(\rho_{\text{ref},RB,\text{ext}} \otimes \rho_{\text{ref}',R'B',\text{ext}'} \\ &\quad \times U_{RB}^\dagger \otimes U_{R'B'}^\dagger \mathcal{S}_{R;R'} U_{RB} \otimes U_{R'B'} \mathcal{S}_{\text{ext};\text{ext}'}) \end{aligned} \quad (12)$$

where $\mathcal{S}_{A;A'}$ is the swap operator between subsystems A and A' , similarly, $\mathcal{S}_{AB;A'B'} = \mathcal{S}_{A;A'} \mathcal{S}_{B;B'}$. Then the average over the Haar measure is accomplished by an application of Schur's lemma [5]

$$\begin{aligned} &\langle\langle U_A^\dagger \otimes U_{A'}^\dagger \mathcal{S}_{A_2;A_2'} U_A \otimes U_{A'} \rangle\rangle \\ &= \frac{A_2(A_1^2 - 1)}{A^2 - 1} \mathbb{1}_{A;A'} + \frac{A_1(A_2^2 - 1)}{A^2 - 1} \mathcal{S}_{A;A'}. \end{aligned} \quad (13)$$

This approach allows us to straight-forwardly compute

the required purities as

$$\begin{aligned}
p(\text{ref}) &= \frac{1}{K}, \quad p(\text{ext}) = \frac{1}{N}, \quad p(\text{ref,ext}) = \frac{1}{KN}, \\
p(R) &= \frac{1}{(RB)^2 - 1} \left(R(B^2 - 1) + \frac{B(R^2 - 1)}{KN} \right), \quad (14) \\
p(R, \text{ext}) &= \frac{1}{(RB)^2 - 1} \left(\frac{R(B^2 - 1)}{N} + \frac{B(R^2 - 1)}{K} \right),
\end{aligned}$$

with $p(B, \text{ext})$ and $p(B, \text{ext})$ given by the above expressions under the exchange $R \leftrightarrow B$, similarly the exchange $K \leftrightarrow N$ gives us expressions for $p(\text{ref}, R)$, etc.

-
- [1] D. N. Page, Information in black hole radiation. Phys. Rev. Lett. **71**, 3743-3746 (1993).
 - [2] C. H. Bennett and S. J. Wiesner, Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. Phys. Rev. Lett. **69**, 2881-2884 (1992).
 - [3] P. Hayden and J. Preskill, Black holes as mirrors: quantum information in random subsystems. JHEP **2007**(09), 120 (2007).
 - [4] G. 't Hooft, Dimensional reduction in quantum gravity. in *Salamfestschrift: A collection of talks*, Vol. 4, ed. A. Ali, J. Ellis and S. Randjbar-Daemi (World Scientific, Singapore, 1993).
 - [5] A. Abeyesinghe, I. Devetak, P. Hayden and A. Winter, The mother of all protocols: Restructuring quantum information's family tree. arXiv:quant-ph/0606225.
 - [6] J. Eisert, M. Cramer and M. B. Plenio, Area laws for the entanglement entropy – a review. Rev. Mod. Phys. to appear, arXiv/0808.3773.
 - [7] G. 't Hooft, On the quantum structure of a black hole. Nucl. Phys. B **256**, 727-745 (1985).
 - [8] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, A quantum source of entropy for black holes. Phys. Rev. D **34**, 373-383 (1986).
 - [9] M. Srednicki, Entropy and area. Phys. Rev. Lett. **71**, 666-669 (1993).
 - [10] S. Hawking, J. Maldacena and A. Strominger, DeSitter entropy, quantum entanglement and AdS/CFT. JHEP **2001**(05), 001 (2001).
 - [11] R. Brustein, M. B. Einhorn and A. Yarom, Entanglement interpretation of black hole entropy in string theory, JHEP **2006**(01), 098 (2006).
 - [12] R. Emparan, Black hole entropy as entanglement entropy: a holographic derivation. JHEP **2006**(06), 012 (2006).
 - [13] S. L. Braunstein and A. K. Pati, Quantum information cannot be completely hidden in correlations: Implications for the black-hole information paradox. Phys. Rev. Lett. **98**, 080502 (2007).
 - [14] D. Kretschmann, D. Schlingemann, and R. F. Werner, A continuity theorem for Stinespring's dilation. J. Funct. Analysis, **255**, 1889-1904 (2008).
 - [15] D. W. Leung, Quantum Vernam cipher. Quantum Inf. Comput. **2**, 14 (2001).
 - [16] L. Susskind, L. Thorlacius and J. Uglum, The stretched horizon and black hole complementarity. Phys. Rev. D **48**, 3743-3761 (1993).
 - [17] I. Bengtsson and K. Zyczkowski, Geometry of Quantum States: An Introduction to Quantum Entanglement. (Cambridge University Press, Cambridge, 2006).
 - [18] P. A. Mello, Averages on the unitary group and applications to the problem of disordered conductors. J. Phys. A **23**, 4061-4080 (1990).